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LETTER TO THE EDITOR

A note on the symmetries of the 3*j*-coefficient

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Abstract. It is shown that a set of six series representations, and correspondingly a set of six ${}_3F_2(1)$'s, is necessary and sufficient to account for the 72 symmetries of the 3*j*-coefficient.

It has long been established (Wigner 1940, reprinted in Biedenharn and Van Dam 1965) that the 3*j*-coefficient defined by:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta(m_1+m_2+m_3)(-1)^{j_1-j_2-m_3}\Delta(j_1j_2j_3)\prod_{i=1}^{3}[(j_i+m_i)!(j_i-m_i)!]^{1/2}$$
$$\times\sum_t(-1)^t\left(t!\prod_{k=1}^{2}(t-\alpha_k)!\prod_{l=1}^{3}(\beta_l-t)!\right)^{-1}, \qquad (1)$$

where

$$\max(\alpha_1,\alpha_2)\leqslant t\leqslant\min(\beta_1,\beta_2,\beta_3),$$

$$\beta_1=j_1-m_1, \qquad \beta_2=j_2+m_2, \qquad \beta_3=j_1+j_2-j_3,$$

$$\alpha_1=j_1-j_3+m_2=j_1-m_1-(j_3+m_3), \qquad \alpha_2=j_2-j_3-m_1=j_2+m_2-(j_3-m_3)$$

and

$$\Delta(xyz)=[(x+y-z)!(x-y+z)!(-x+y+z)!/(x+y+z+1)!]^{1/2},$$

is invariant to permutations of the three angular momenta (column permutations) and to space reflection ($m_i\to-m_i$), thereby exhibiting the classical 12-element symmetry group.

Regge (1958) arranged the nine integer parameters referred to by Racah (1942) into a 3×3 square symbol to represent the 3*j*-coefficient as:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left\Vert\begin{matrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{matrix}\right\Vert$$
$$=\Vert R_{ik}\Vert, \qquad (2)$$

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and discovered that there exists a 72-element Regge symmetry group, comprising the classical symmetry group already mentioned, for the $3j$-coefficient, due to the invariance of $\|R_{ik}\|$ to its column and row permutations and a reflection about its diagonal. The fact that all sums by columns and rows are equal to $J=j_1+j_2+j_3$, leads to the following nine relations among the elements of $\|R_{ik}\|$:

$$R_{lp}+R_{mp}=R_{nq}+R_{nr}, \tag{3}$$

for cyclic permutations of both $(l\,m\,n)=(1\,2\,3)$ and $(p\,q\,r)=(1\,2\,3)$.

Racah (1942) has shown that assuming the argument of one of the five factorials in (1) as the summation index instead of t leads to some symmetry properties of the $3j$-coefficient. Making such a substitution successively for each of the five factorials in (1) it is possible to obtain five series representations. These five series representations together with (1), which is the only series conventionally given in literature (e.g. Smorodinskii and Shelepin 1972), constitute a set of six series representations, given below, for the $3j$-coefficient, which is necessary and sufficient to account for the 72 symmetries. Since the Regge symmetry group is evident when the $3j$-coefficient is represented by $\|R_{ik}\|$, we define the set of six series representations, in the following compact notation, as:

$$\begin{aligned}\|R_{ik}\|=\delta(m_1+m_2+m_3)&\prod_{i,k=1}^{3}[R_{ik}!/(J+1)]^{1/2}\\&\times(-1)^{\sigma(pqr)}\sum_s(-1)^s[s!(R_{2p}-s)!(R_{3q}-s)!(R_{1r}-s)!\\&\times(s+R_{3r}-R_{2p})!(s+R_{2r}-R_{3q})!]^{-1},\end{aligned} \tag{4}$$

for all six permutations of $(p\,q\,r)=(1\,2\,3)$, with

$$\sigma(pqr)=\begin{cases}R_{3p}-R_{2q}, & \text{for even permutations,}\\ J+R_{3p}-R_{2q}, & \text{for odd permutations.}\end{cases}$$

It can be shown that this set of six series representations can also be obtained by permuting the indices (1 2 3) in equation (1) and remembering that the series acquires an additional phase factor $(-1)^J$ for odd permutations.

The six column permutations of $\|R_{ik}\|$ are in one-to-one correspondence with the six series representations, thereby spanning the whole set given by equation (4). Each series representation exhibits 12 of the 72 distinctly different symmetries of the $3j$-coefficient and this 12-element symmetry group is isomorphic to the product of permutation groups of three objects (R_{2p}, R_{3q}, R_{1r}) and two objects $(R_{3r}-R_{2p}, R_{2r}-R_{3q})$.

Regge (1958) symmetries for the $3j$-coefficient can be written down explicitly as†:

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}
&=\begin{pmatrix} j_1 & \tfrac12(j_2+j_3+m_1) & \tfrac12(j_2+j_3-m_1)\\ j_2-j_3 & \tfrac12(j_3-j_2+m_1)+m_2 & \tfrac12(j_3-j_2+m_1)+m_3\end{pmatrix}\\
&=\begin{pmatrix} \tfrac12(j_1+j_3+m_2) & j_2 & \tfrac12(j_1+j_3-m_2)\\ \tfrac12(j_3-j_1+m_2)+m_1 & j_1-j_3 & \tfrac12(j_3-j_1+m_2)+m_3\end{pmatrix}\end{aligned}$$

† Surprisingly, these are not available in this form in literature (e.g. Biedenharn and Van Dam 1965).

$$
\begin{aligned}
&= \begin{pmatrix} \frac{1}{2}(j_1+j_2-m_3) & \frac{1}{2}(j_1+j_2+m_3) & j_3 \\ \frac{1}{2}(j_1-j_2+m_3)+m_1 & \frac{1}{2}(j_1-j_2+m_3)+m_2 & j_2-j_1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}(j_1+j_2-m_3) & \frac{1}{2}(j_2+j_3-m_1) & \frac{1}{2}(j_1+j_3-m_2) \\ j_3-\frac{1}{2}(j_1+j_2+m_3) & j_1-\frac{1}{2}(j_2+j_3+m_1) & j_2-\frac{1}{2}(j_1+j_3+m_2) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}(j_1+j_2+m_3) & \frac{1}{2}(j_2+j_3+m_1) & \frac{1}{2}(j_1+j_3+m_2) \\ \frac{1}{2}(j_1+j_2-m_3)-j_3 & \frac{1}{2}(j_2+j_3-m_1)-j_1 & \frac{1}{2}(j_1+j_3-m_2)-j_2 \end{pmatrix}.
\end{aligned} \tag{5}
$$

The 12 symmetries exhibited by a given series representation arise due to: (i) the combined operation of an odd column permutation and the space reflection; and (ii) a Regge symmetry given above, or, a Regge symmetry on which an even column permutation is superposed, or, a Regge symmetry on which is superposed a combined odd column permutation and the space reflection.

From these observations it follows that the set of six series representations is necessary and sufficient to account for the known symmetries of the $3j$-coefficient. Conventional ways (deShalit and Talmi 1966) of establishing the symmetry relations are tedious and some (Rose 1957) necessarily resort to a substitution for the summation index. Our definition of $\|R_{ik}\|$ as a set of six series representations enables one to establish the symmetries more easily.

In terms of generalised hypergeometric functions of unit argument, equation (4) can be re-arranged into:

$$
\begin{aligned}
\|R_{ik}\| = \delta(m_1+m_2+m_3) \prod_{i,k=1}^{3} [R_{ik}!/(J+1)]^{1/2}(-1)^{\sigma(pqr)} \\
\times(\Gamma(1-A, 1-B, 1-C, D, E))^{-1}\, {}_3F_2(A, B, C; D, E; 1),
\end{aligned} \tag{6}
$$

where

$$
A=-R_{2p}, \qquad B=-R_{3q}, \qquad C=-R_{1r}, \qquad D=1+R_{3r}-R_{2p},
$$
$$
E=1+R_{2r}-R_{3q} \qquad \text{and} \qquad \Gamma(x, y, \ldots)=\Gamma(x)\Gamma(y)\ldots,
$$

for all permutations of $(p\,q\,r)=(1\;2\;3)$. This set of six ${}_3F_2(1)$'s derived straightforwardly from the set of series representations, are all of the van der Waerden's form, satisfying the condition:

$$
D+E-A-B-C=j_1+j_2+j_3+2,
$$

which is called as the characteristic sum of the ${}_3F_2(1)$'s by Huszar (1972). The ${}_3F_2(1)$ corresponding to $(p\,q\,r)=(1\;2\;3)$ is given by equation (5.21) of Smorodinskii and Shelepin (1972). It is to be noted that in the work of D'Adda *et al* (1974), on symmetries of extended $3j$-coefficients, a set of real variables are introduced to express the $3j$-coefficient of SU(2) in terms of entire functions proportional to the hypergeometric functions, ${}_3F_2(1)$'s.

The relation:

$$
\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} l & \frac{1}{2}l & \frac{1}{2}l \\ 2l'-l & \frac{1}{2}l-l' & \frac{1}{2}l-l' \end{pmatrix} \tag{7a}
$$

referred to by Morgan (1977), is a direct consequence of the first of the Regge symmetries given by equations (5). It is interesting to note that, in fact, the Regge symmetries also yield:

$$\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} l-\frac{1}{2}l' & l' & l-\frac{1}{2}l' \\ -\frac{1}{2}l' & l' & -\frac{1}{2}l' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(l+l') & \frac{1}{2}(l+l') & l-l' \\ \frac{1}{2}(l-l') & \frac{1}{2}(l-l') & l'-l \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(l+l') & \frac{1}{2}l & l-\frac{1}{2}l' \\ \frac{1}{2}(l-3l') & \frac{1}{2}l & \frac{1}{2}(3l'-2l) \end{pmatrix}. \tag{7b}$$

Recently, Lockwood (1976) introduced a set of parameters: n, a, b, c, d and e, for the $3j$-coefficient, which when explicitly written down are just the parameters of $\|R_{ik}\|$ or differences between two of them. The conclusion of Lockwood—that there exists a nine-to-one homomorphism which reduces the known 72-element symmetry group to an eight element group—is based on his erroneous observation that the series given by equation (4) in his article exhibits only a 4-element symmetry group, denoted by:

$$(3J)(n; a, b; c, d), \tag{8}$$

since this series, as well as the numerical and phase factors (P and T, respectively), exhibit the 12-element symmetry group:

$$(3J)(a, b; n, n+c, n+d), \tag{9}$$

where a and b can be identified with $(R_{3r}-R_{2p})$ and $(R_{2r}-R_{3q})$, while $n, n+c$ and $n+d$ can be identified with R_{2p}, R_{3q} and R_{1r}. Further, the ordering of Lockwood's triples (p, q, r) and (f, g, h)—which correspond to $(\beta_1, \beta_2, \beta_3)$ and $(\alpha_1, \alpha_2, 0)$, respectively—can be shown to lead only to the set of six series representations given by our equations (4). Therefore, the nine parameters in $\|R_{ik}\|$ are the canonical parameters for the $3j$-coefficient and not the one's introduced by Lockwood, since it is in terms of $\|R_{ik}\|$ that we can account for the Regge symmetry group of the $3j$-coefficient.

Finally, a computer program based on the set of six ${}_3F_2(1)$'s has been shown (Srinivasa Rao and Venkatesh 1977) to be not only efficient but also faster, at best by 5–15%, than the best available program due to Wills (1971).

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